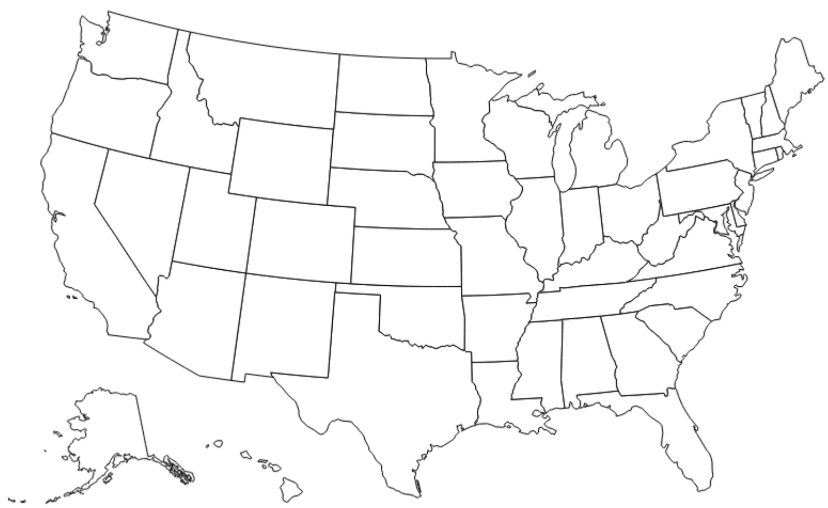
Grade 9/10 Math Circles

February 22

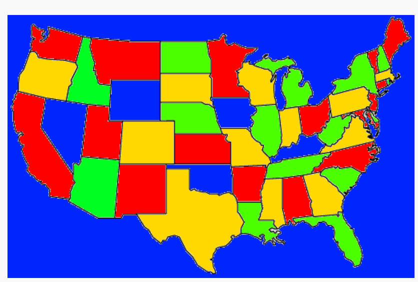
The Shape of You - Solutions Set

Graph Colouring

1. Try to color the map of the United States so that no two neighbouring states have the same color, **using at most four colors**. Note that coloring maps is the same as coloring graphs (why?)



Solution: Here's one possible solution (it even colors the surrounding body of water):



Notice that coloring maps is the same as coloring graphs because we can associate each "region" in a map with a dot and draw lines between any two regions that are beside each other to get a planar graph.

2. Prove the **Handshake Lemma** from the lecture notes.

Solution: Recall that we define the degree of a dot to be the number of lines that are attached to it. So this Lemma just boils down to the fact that every line joins exactly *two* dots together. That means if we sum up the degrees of all the dots, we are double-counting every line, meaning we end up with twice the number of lines.

This result is named the "Handshake Lemma" because you can view each dot as a person and each line as a handshake between two people. Counting the number of handshakes that each person made gives you exactly twice the total number of handshakes.

Series Examples

3. Prove the **Geometric series** formula from class:

$$1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}$$

Solution: Let's define

$$S = 1 + r + r^2 + r^3 + \ldots$$

so that our goal is to find a different equality for the value of S, whatever this value is. Notice that if we multiply by r, we somehow "shift" the terms in this sum:

$$rS = r + r^2 + r^3 + \ldots$$

Subtracting term-term-by-term, almost everything cancels, and we get

$$S - rS = 1 \implies (1-r)S = 1 \implies S = \frac{1}{1-r}$$

This proof is a bit shaky because subtracting off an "infinite number of terms" isn't automatically well-defined (unless you can prove it!) but it suffices for our purposes.

The (much) better way to show this is to use the same technique to figure out a formula for the partial sums

$$S_n = 1 + r + r^2 + \ldots + r^n = \frac{1 - r^{n+1}}{1-r}$$

(try it yourself!) and then use a **limit** of n to infinity, ie. let n get arbitrarily large. If $|r| < 1$, then $r^{n+1} \to 0$, so we end up getting the formula for S above. This is the better proof because S_n is a *finite* sum and we already have a good, solid understanding of how to add finitely many things together. You have to be very careful when working with infinity, or you might find some weird results and paradoxes.

4. What should the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = ?$$

equal? Is it finite or infinite?

Solution: This sum, called the **Harmonic series**, is actually infinite! We'll show this by making each term (possibly) smaller, so that the smaller sum will obviously be infinity.

$$1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{\geq \frac{1}{4}+\frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{\geq \frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}} + \ldots$$

Keep doing this comparison by grouping together terms so that their sum is at least $\frac{1}{2}$. So our original sum is at least as big as

$$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

and this is clearly infinite. No finite number is larger than infinity, therefore our original sum must also be infinite.

5. What should the sum

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = ?$$

equal? Is it finite or infinite?

Solution: Unlike the previous sum, this sum turns out to be finite! We say that it **converges** (infinite sums are treated in calculus very formally like limits, and the definition of limit convergence is a strictly defined concept, even though in these solutions I seem to be "hand-waving" it a bit just to show you some cool/weird math that you'll study later on).

The proof that this sum is finite is beyond the scope of this Math Circles session, but it turns out that

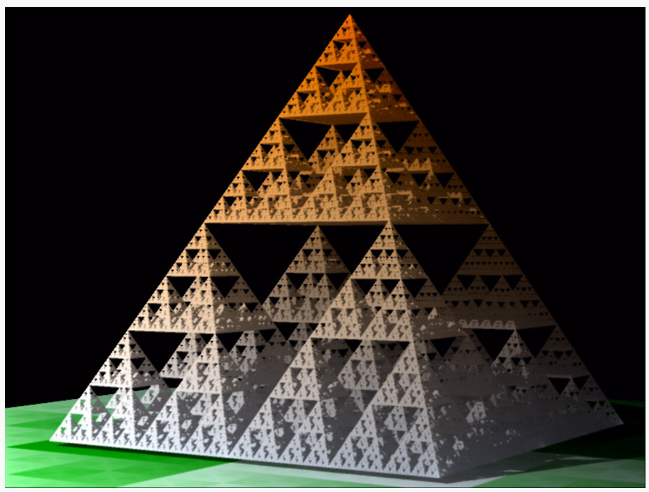
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \ln(2)$$

where $\ln(2)$ stands for the **natural logarithm** of 2, and has a value approximately 0.69314718056 (check this yourself by adding up the first couple of terms and seeing if the partial sums get close to this number!)

Three-Dimensional Fractals

6. Try drawing (or building) Sierpinski's Gasket in three dimensions!

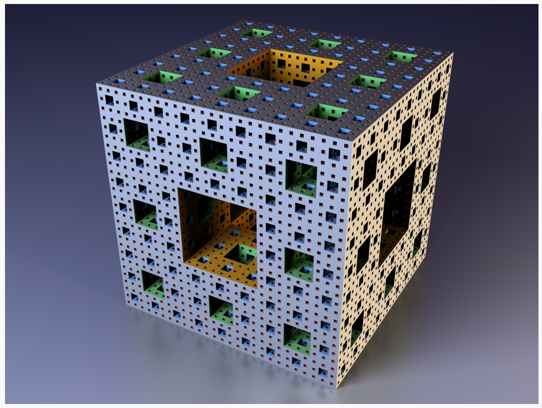
Solution: Here's a computer-generated visual:



If you can draw the three dimensional shadow of a four dimensional Sierpinski's Gasket, I'd be impressed.

7. Try drawing (or building) Sierpinski's carpet in three dimensions!

Solution: Here's a computer-generated visual:



For funsies, you can build your own Menger Sponge out of business cards and origami at home! (Try it, I did). In 2014, mathematicians Matt Parker and Laura Taalman began a project to build the largest fractal in the world (made up of giant pieces that were built all around the world) using origami. You can check out the project and print out the business cards yourself at https://www.megamenger.com/, it's very cool. Send me a picture to my email **mgusak@uwaterloo.ca** if you're able to build this!